Crossover behavior in a mixed-mode fiber bundle model

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We introduce a mixed-mode load sharing scheme in a fiber bundle model. This model reduces exactly to equal-load-sharing (ELS) and local-load-sharing (LLS) models at the two extreme limits of a single-load-sharing parameter. We identify two distinct regimes: (a) the mean-field regime where the ELS mode dominates and (b) the short-range regime dominated by the LLS mode. The crossover behavior is explored through a numerical study of the strength variation, the avalanche statistics, susceptibility and relaxation time variations, the correlations among the broken fibers, and their cluster analysis. Analyzing the moments of the cluster size distributions we locate the crossover point of these regimes. We thus conclude that even in one dimension, the fiber bundle model shows crossover behavior from mean-field to short-range interactions.

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I. INTRODUCTION

The fiber bundle model represents a simple, stochastic fracture-failure process [1] in materials subjected to external load. The model consists of three basic ingredients: (a) a discrete set of N elements located at sites of a lattice, (b) a probability distribution of the strength threshold of individual elements (fibers), and (c) a load-transfer rule which distributes the terminal load carried by the failed fibers to the surviving fibers. The model study was initiated by Peirce [2] in the context of testing the strength of cotton yarns. Since then, this model has been studied and modified by many groups [3–25] using analytic as well as numerical methods. Fiber bundles are of two classes with respect to the time dependence of the fiber strength threshold: "Static" bundles contain fibers whose threshold strengths are independent of time and such bundles are subjected to quasistatic loading; i.e., the load is increased steadily up to the complete failure of the bundles. The load or stress σ (load per fiber) is an independent variable here, and the strength of the bundle is determined by the maximum value of the applied load or stress (σ_c) that can be supported by the bundle. On the other hand, "dynamic" bundles consist of fibers having time-dependent strength and the fibers fail due to fatigue [4–7] after a period of time which varies fiber to fiber. The time taken for complete failure is called the lifetime of the bundle. According to the load sharing rule, fiber bundles are being classified into two groups: equal-load-sharing (ELS) bundles [8–17] or democratic bundles and local-load-sharing (LLS) bundles [18–20]. In the ELS models all the intact fibers equally share the terminal load of a failed fiber, whereas in LLS model the terminal load gets shared among the intact nearest neighbors. ELS models show a phase transition from partial failure to total failure at a critical strength (σ_c). The critical behavior in the failure dynamics of ELS bundles has been solved analytically [15,16] and the universality of the ELS model has been established [17] recently. However, the strength of LLS models goes to zero [21–23] at the limit of infinite system size and this does not permit any critical behavior in the failure process.

The ELS and LLS models belong to two opposite extremes with respect to the spatial correlations in stress redistributions. These models do not incorporate any type of stress gradient among the intact fibers which is an usual expectation. Therefore a load sharing scheme in between ELS and LLS should be a realistic approach to study the failure of heterogeneous materials. Hansen and Hemmer [24] introduced a "λ model" to interpolate between ELS and LLS models where λ is an adjustable stress-transfer factor. Although they conjectured the existence of a critical crossover value λ_c which separates the mean-field (ELS) regime and the short-range (LLS) regime, what would be the exact crossover point was not answered. A recent approach by Hidalgo *et al.* [25] incorporates both the ELS and LLS modes introducing an effective range of interaction parameter (γ) which is actually the power of the stress redistribution function. They observed crossover behavior in the strength variation and in the avalanche statistics of the failures. Also they determined the crossover point (γ_c) through moment analysis of the cluster size distributions before total failure.

In this paper we develop a mixed-mode load sharing (MMLS) model which interpolates the ELS and LLS models correctly. We intend to study whether this model shows a continuous transition from mean-field (ELS) behavior to extreme statistics (LLS) or if there exists a definite crossover point.

We organize this paper as follows: After this introduction (Sec. I) we present our MMLS model in Sec. II. Section III contains observations of crossover behavior through a numerical study of the model. The analysis to determine the exact crossover point is given in Sec. IV. The final section (Sec. V) is devoted to a discussions including our conclusions.

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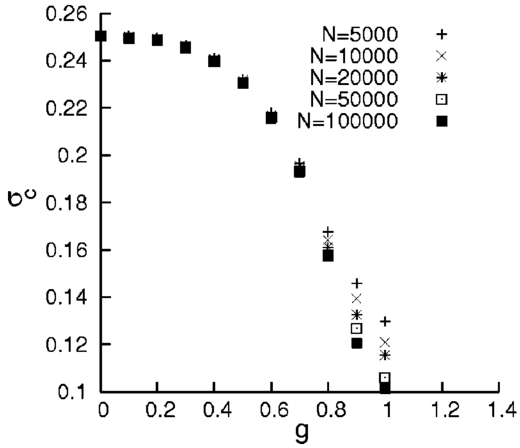

FIG. 1. The strength of the bundle for different system sizes (N) as a function of the weight parameter g.

II. MODEL

Our mixed-mode load sharing scheme is basically a coupling of ELS and LLS modes: When a fiber fails, a fraction (g) of its terminal load gets shared among the nearest neighbors of the failed fiber (LLS rule) and the rest ($1-g$ fraction) is distributed equally among all the surviving fibers (ELS rule). Here g is the weight parameter of the MMLS scheme. Therefore, the model reduces exactly to the ELS model for $g=0$, and for $g=1$, it becomes a pure LLS one. As we have chosen a $(1-D)$-fiber-bundle model (with periodic boundary condition), the number of nearest neighbors is always 2. We study the behavior of the model for the entire range $0 \leqslant g \leqslant 1$ using Monte Carlo simulations for stepwise equal-load increments [15–17] until total failure of the bundle. During the entire study we consider a uniform (on average) distribution of the fiber strength threshold in the bundle.

III. CROSSOVER BEHAVIOR

A. Strength of the bundle

It has been known since Daniels [3] that the ELS bundles have a nonzero strength (σ_c) above which the bundle fails completely. Recently it has been shown analytically [15,16] that for a uniform fiber threshold distribution, the bundle's strength approaches the value 1/4 as the system size goes to infinity. On the other hand, LLS bundles do not have any nonzero strength [21–23]. In our MMLS model we intend to study the strength variation of bundles with system size as well as with the weight parameter g.

As g increases, the bundle becomes weaker due to the short-range (LLS) interactions. Therefore σ_c decreases with increasing g values (Fig. 1). We can see that σ_c seems to be independent of the system size (dominance of ELS) up to $g=0.7$, and beyond $g=0.8$, a strong system size dependence (dominance of LLS) appears. This observation is supported by Fig. 2, where we have shown the logarithmic size dependence of σ_c. Up to $g=0.7$, the curves eventually become flat as the system size increases. But for $g \geqslant 0.8$, all the curves fall (following an inclined straight line). Thus the two regimes are differentiated clearly.

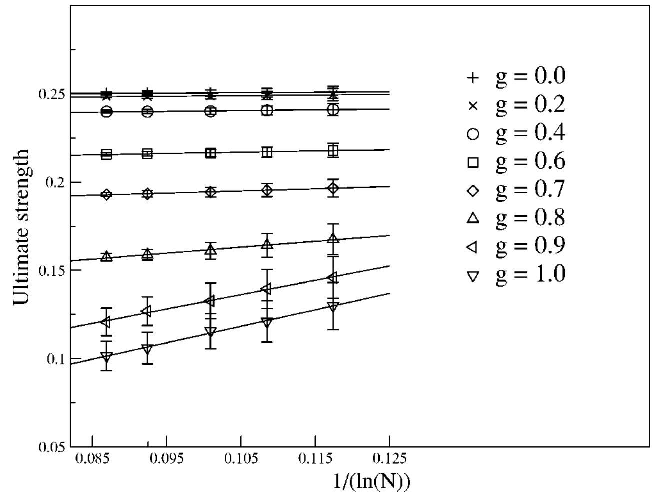

FIG. 2. The logarithmic size dependence of bundle's strength for different values of g. The straight lines represent the best fit.

B. Avalanche size distribution

The avalanche size distribution characterizes the fracture process by reflecting the precursory activities toward complete failure. This can be related to the acoustic emissions observed in material failure [26–28]. Hemmer and Hansen showed [8] analytically that for ELS models the avalanche size distribution follows a universal power law with exponent value $-5/2$. But for LLS models the numerically estimated apparent exponent value is quite larger 4.5 [9]. Later it was shown analytically (for flat threshold distributions) that for the LLS model, no universal power-law asymptotics exists [10].

Here we have measured (Fig. 3) the avalanche size distributions for different g values. Clearly, two groups of curves

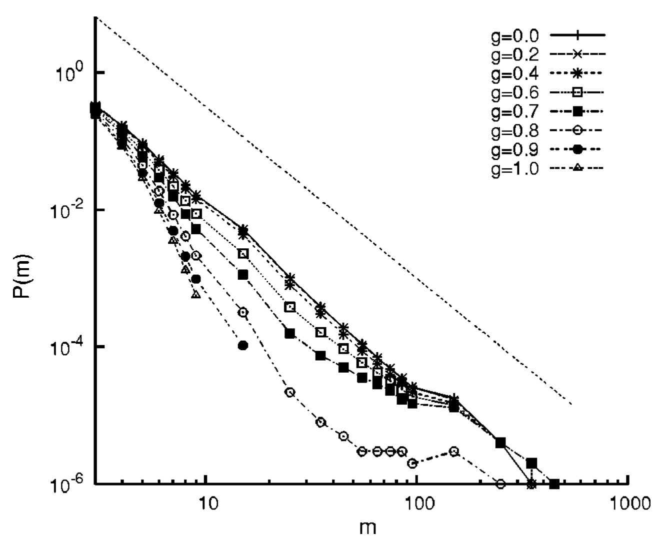

FIG. 3. Avalanche size distribution for different values of the weight parameter g (averaging over 5000 configurations for system size $N=20\,000$). The dotted line represents the mean-field result having exponent value $-5/2$. Clearly, the upper group of curves can be fitted by the mean-field power law whereas the lower group does not show power law at all.

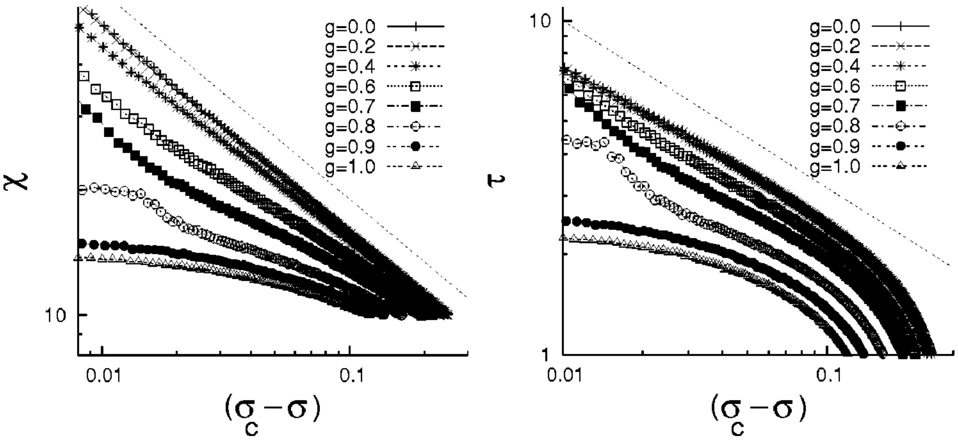

FIG. 4. The susceptibility (χ) and relaxation time (τ) variations for different g values. The bundle contains 10 000 fibers and the data are averaged over 10 000 configurations.

appear. The upper group ($0 \leqslant g \leqslant 0.7$) can be fitted with the mean-field result ($-5/2$) where as the lower group ($0.8 \leqslant g \leqslant 1.0$) shows a clear deviation from the power law.

C. Susceptibility and relaxation time variations

Recently, the dynamic response parameters, susceptibility (χ) [13–16,29], and relaxation time (τ) [15,16] have been studied in fiber bundle models. The susceptibility is defined as the number of fibers fail due to an infinitesimal change of the external stress (σ) on the bundle and the relaxation time is the time (number of stress redistributions) the bundle takes to come to a stable fixed point at an external stress (σ). For the ELS model, the susceptibility and relaxation time seem to follow a power law with the applied stress and both of them diverge [13,15–17] at the critical strength σ_c: $\chi \sim (\sigma_c - \sigma)^{-1/2}$ and $\tau \sim (\sigma_c - \sigma)^{-1/2}$. However, one cannot expect such scaling behavior in LLS models due to the absence of "critical" strength. The stepwise equal-load increment method [15,16] enables us to measure χ and τ for different values of g (Fig. 4). The power-law behavior (with mean-field exponent $-1/2$) remains unchanged up to $g=0.7$ and for $g \geqslant 0.8$ the curves do not follow power laws at all. Thus the susceptibility and relaxation time variations also suggest a transition from the mean-field to short-range behavior to happen in between $g=0.7$ and $g=0.8$.

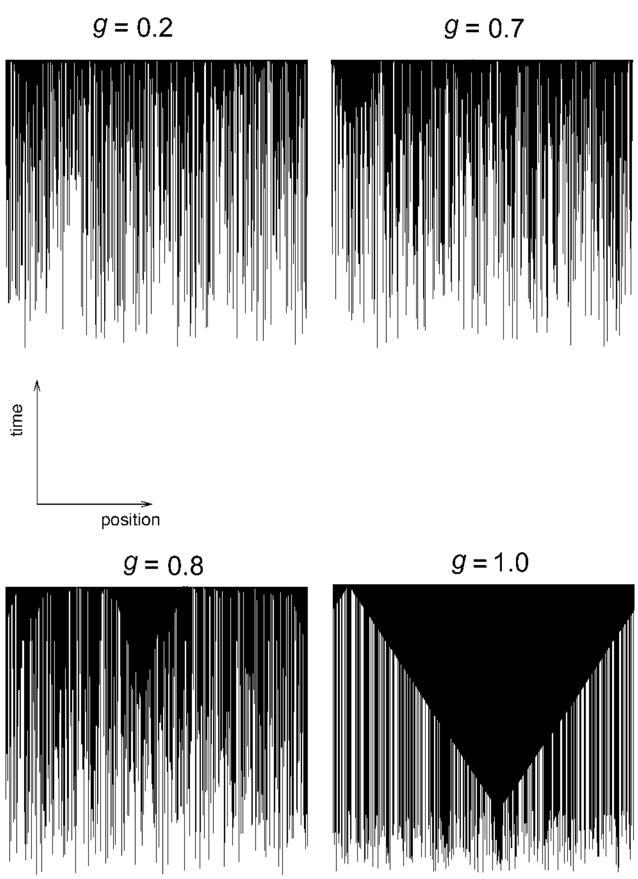

FIG. 5. The space-time diagram of the breakdown sequence in the MMLS model. The positions of the fibers are marked on the x axis, and the y axis is a "time" axis where time indicates the number of stress redistributions starting from initial loading. The white color represents intact fibers while the black regions stand for the broken fibers.

D. Correlations among the broken fibers

The breakdown sequence reflects the correlations of the breaking process [24]. While the ELS model simply ignores

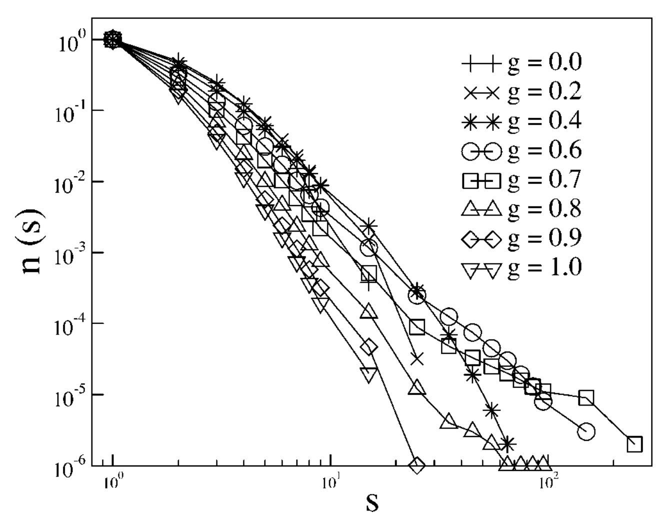

FIG. 6. Cluster size distributions of broken fibers (just before complete failure) for different g values (averaging over 5000 samples for $N=20\,000$).

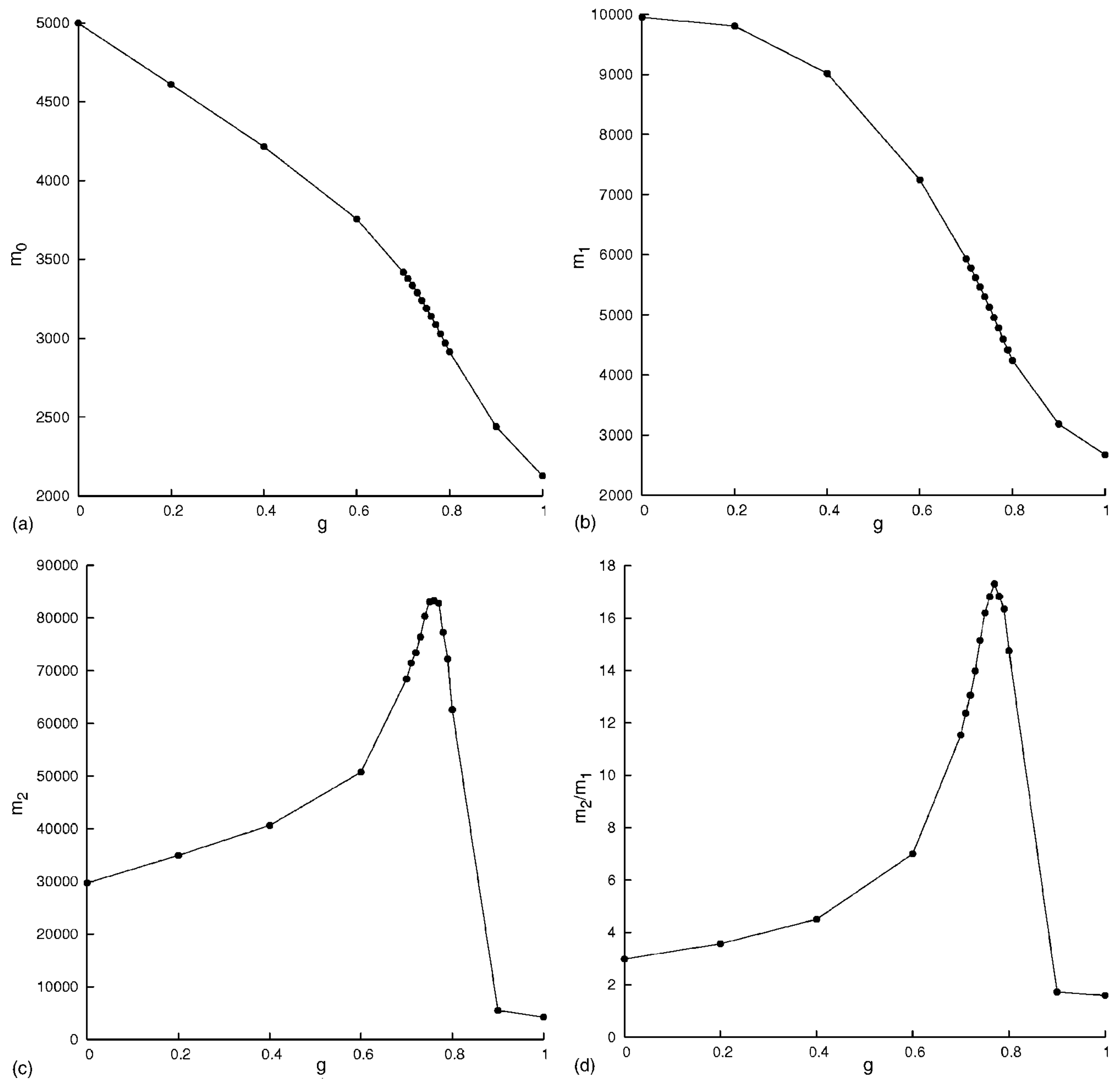

FIG. 7. The moments of the cluster size distributions as a function of the weight parameter g (averaging over 5000 samples for $N = 20\,000$).

the spatial arrangement of the fibers, the LLS model gives much importance to it. Therefore, as g increases (LLS mode dominates) the breaking process becomes more and more correlated (Fig. 5). Here also we can identify two distinct regimes. We cannot see any spatial correlation among the broken fibers (except near total failure) up to $g=0.7$, whereas for $g \geqslant 0.8$ strong correlations (black patch) develop long before total failure.

IV. DETERMINATION OF THE EXACT CROSSOVER POINT THROUGH CLUSTER MOMENT ANALYSIS

The fracture process can also be characterized by analyzing the clusters of broken fibers just before complete failure [12,25]. The size distributions of the clusters $[n(s)$ vs $s]$ are shown (Fig. 6) for different values of g. Although the distributions appear as two groups, it is not possible to identify the exact crossover point from this. Therefore we go for the moment analysis: the kth moment of the cluster distributions is defined [25] as

$$m_k = \int s^k n(s) ds. \tag{1}$$

Clearly the zeroth moment (m_0) gives the total number clusters and the first moment gives the total number of broken fibers. We can get the average cluster size dividing the second moment (m_2) by first moment (m_1).

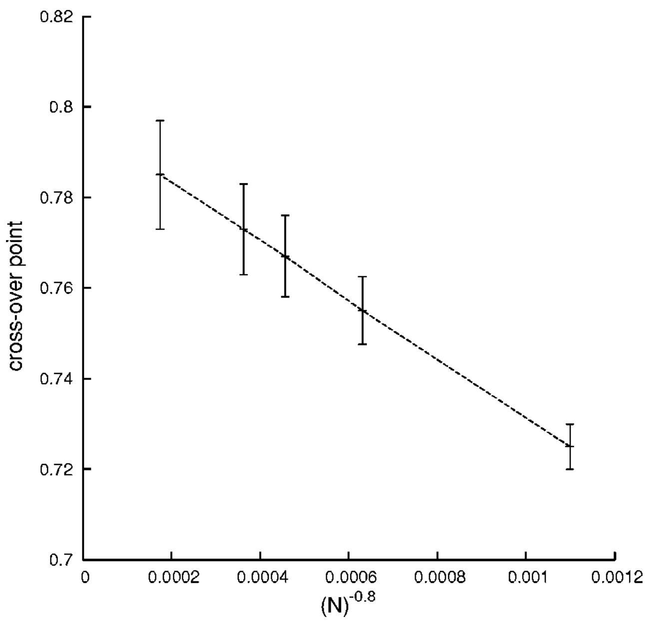

FIG. 8. System size dependence of the crossover point.

In case of the pure ELS mode ($g=0$), we have only long-range interactions and the clusters are randomly distributed within the lattice. As g increases the stress redistribution becomes more and more localized in the neighborhood of the failed fibers and a few isolated crack can trigger the complete rupture through growth and coalescence mechanism. Therefore the pure ELS mode can store the maximum crack (cluster) and this capacity should decrease with the increase of g. We can see (Fig. 7) that both m_0 and m_1 decrease with increasing g value and they fall drastically in between $g=0.7$ and $g=0.8$. This crossover is very robust in the case of m_2 and average cluster size (m_2/m_1), both of which show a sharp peak, which indicates the dominance of the LLS mode over the ELS mode [25].

To check how the crossover point changes its position with system size, we have done a similar cluster moment analysis for several system sizes. We observe a weak system size dependence of the peak position—i.e., the crossover point (Fig. 8). With proper extrapolation we determine the crossover point (g_c) to be at $g=0.79\pm0.01$ for infinite system size.

V. CONCLUSION

The fracture and breakdown of loaded materials is basically a cooperative phenomenon guided by the load redistribution mechanism. Here a "crack" opens up when an element (fiber) fails after external loading. This single-fiber failure should affect the neighbors much than the distant elements (like in electric fuse models [1]). Therefore a high stress concentration (after the load redistribution) around a crack (failed fiber) is a natural expectation. The ELS models do not incorporate any spatial correlations and exhibit perfect democracy (mean field), whereas the LLS models confine themselves within the nearest-neighbor interactions. In this situation attempts [24,25] to study the failure behavior in between ELS and LLS regimes would be most welcome. Also, a recent experiment on loaded wood fiber [30] demands an intermediate-load-sharing scheme to explain the observed strength variation. The "λ model" [24] becomes a LLS model at $\lambda=1$. But it cannot be reduced to a pure ELS model at $\lambda=0$, as the neighbors of the just broken fibers become "immunized" against failure. Although the "variable-range-of-interaction" model [25] determines the exact crossover point, it remains silent about the system size dependence of crossover point, which is nevertheless an important issue.

Our mixed-mode load sharing model exactly reduces to the ELS model at $g=0$ and to the LLS model at $g=1$. We establish numerically that the MMLS model in one dimension shows a distinct crossover behavior from mean-field to short-range interactions. The strength (σ_c) variation of the bundle with system size, the avalanche statistics, and the failure dynamics (susceptibility and relaxation time) suggest that the crossover point (g_c) must be in between $g=0.7$ and $g=0.8$. Cluster size analysis determines the exact crossover point in one dimension for several system sizes and a proper extrapolation suggests the crossover point to be $g_c=0.79\pm0.01$ at the limit of infinite system size. For $g<g_c$ the model exhibits critical behavior (supported by the power laws) for the dominance of the ELS mode. But the fluctuations suppress any critical behavior after $g=g_c$, where extreme statistics [1] dominates. We should mention that as the ultimate strength (σ_c) of the bundle continuously decreases with increasing g value, we cannot exclude the possibility of different critical behavior for $g=0$, $0<g\leqslant g_c$, and $g>g_c$ in higher dimensions, like in case of $2-D$ Ising systems with disorder [31]. Therefore we expect this crossover behavior in the MMLS model to be more prominent in higher dimensions.

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